EXPANSIONS OF ONE DENSITY VIA POLYNOMIALS ORTHOGONAL WITH RESPECT TO THE OTHER.

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Abstract. We expand the Chebyshev polynomials and some of its linear combination in linear combinations of the $q-$Hermite, the Rogers ($q-$utraspherical) and the Al-Salam–Chihara polynomials and vice versa. We use these expansions to obtain expansions of some densities, including $q-$Normal and some related to it, in infinite series constructed of the products of the other density times polynomials orthogonal to it, allowing deeper analysis and discovering new properties. On the way we find an easy proof of expansion of the Poisson–Mehler kernel as well as its reciprocal. We also formulate simple rule relating one set of orthogonal polynomials to the other given the properties of the ratio of the respective densities of measures orthogonalizing these polynomials sets.

1. Introduction

The aim of this paper is to formulate a simple rule of expanding one density in terms of products of the other density times the polynomial orthogonal with respect to this density. Then to present some of its consequences and applications. The original aim of such expansions was to use them to find some 'easy to generate', simple densities that bound from above other densities that were given in the form of infinite products. In other words the original aim of such expansions was practical and connected with the idea of generating i.i.d. sequences of observations drawn from distributions given by the densities that have difficult to analyze form, e.g. are given in the form of an infinite product. Later however, it turned out that such expansions are interesting by its own allowing deeper insight into distributions that are defined by the densities involved. In particular 'two lines proofs' are possible of the identities that traditionally are proved on a half or more pages.

A simple reflection leads to the conclusion that we deal with this type of situation in the case of e.g. the Poisson–Mehler expansion formula or recently obtained (see [8]) expansion of the $q-$Normal density in terms of products of the Wigner density times appropriately scaled Chebyshev polynomials. Thus it is the time to generalize it, formulate general rule and obtain some new expansions. It will turn out that following this general rule, the difficulty of obtaining expansion of the type discussed in the paper is shifted to difficulties in getting the so called "connection coefficients"

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obtained by expanding one family of orthogonal polynomials with respect to the other.

In particular we will obtain new expansions of the so called $q-$Conditional Normal density in the series of Kesten–McKay type density times some special combinations of Chebyshev polynomials, reciprocals of the Poisson–Mehler expansion formula and expansions of some other more specialized densities.

The two mentioned above densities and distributions defined by them appeared recently in works of Bożejko at al. [6] in the noncommutative or Bryc [2], [3] in the classical probability context. These densities were originally defined in terms of infinite products and thus it was difficult to work with them for someone not familiar with notation, notions and basic results in the so called $q-$series theory.

In many branches or applications of functional analysis such as theory of linear operators or quantum groups we deal with Mercer's type kernels i.e. expressions of the type $\sum_{n\geq 0}r_n\phi_n\left(x\right)\psi_n\left(y\right),$ where $\left\{r_n\right\}_{n\geq 0}$ is a sequence of reals, $\left\{\phi_n\left(x\right)\right\}_{n\geq 0}, \left\{\psi_n\left(x\right)\right\}_{n\geq 0}$ are sequences of square integrable functions from some space $L_2\left(\mathbb{R},\mathcal{B},\gamma\right).$ The problem is to provide conditions for non-negativity of such a kernel when (x,y) belong to some Cartesian product of intervals, prove this non-negativity and also express such a kernel in some compact, easier to analyze, form, i.e. sum it. The point is that many of the expansions that we have obtained in the paper are in fact certain kernels. Variable y plays a rôle of a parameter. By the nature of the expansion we know its sum and know that it is nonnegative. Hence the paper can be helpful solving important problems associated with summing and examining positivity of kernels.

The ideas we are presenting here are universal and can be applied to any densities and systems of orthogonal polynomials.

The paper is organized as follows. The next Section 2 presents general idea of expansion, the main subject of the paper, as well as simple Proposition presenting relationship between sets of polynomials given the ratio of the densities of measures orthogonalizing these sets of polynomials. The task, in a sense, inverse to the idea of expansion. This section contains also Subsection 2.1 that presents some, instructive and believed to be interesting, examples of expansions between simple measures orthogonalizing well known sets of polynomials such as Chebyshev, Hermite and some of their combinations. Next we introduce notation used in the $q-$series theory in Subsection 3.1. Then we list densities that we will analyze in Subsection 3.2. Finally we present families of orthogonal polynomials that will be used in the sequel and associate them with measures that make these sets of polynomials orthogonal in Subsection 3.3. In particular we present here the $q-$Hermite, the Al-Salam–Chihara and the Rogers ($q-$utraspherical) polynomials. The next Section 4 is devoted to listing known and finding some new connection coefficients between considered in the paper families of polynomials. Section 5 presents main results of the paper, that is expansions of one density in the series of the other density times series of polynomials orthogonal with respect to this other measure. We also give some (by no means all possible) immediate consequences that lead to interesting identities. Finally Section 6 contains some lengthy proofs of some of the results of Section 4. This section, following suggestion of the referee, contains also a few sentences presenting basic properties of orthogonal polynomials as well as reference to some literature dedicated to the theory of orthogonal polynomials.

2. Idea of expansion

The idea of expansion that we are going to pursue is general, simple and is not new (it can be found in e.g. in [12], Exercise 2.9). We believe that it is very fruitful and has not been sufficiently exploited. It is as follows.

Suppose we have two measures α and β defined on $\mathbb{R}$. Let us define two spaces $L_2(\mathbb{R},\mathcal{B},\alpha)$ and $L_2(\mathbb{R},\mathcal{B},\beta)$, where $\mathcal{B}$ denotes a set of Borel subsets, of real functions defined on $\mathbb{R}$, square integrable with respect to measures α and β respectively. Assume also that $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$. Further suppose that we know the sets of polynomials $\{a_n(x)\}_{n\geq 0}$ and $\{b_n(x)\}_{n\geq 0}$ defined on $\mathbb{R}$ that are orthogonal with respect to the measures α and β respectively. That is, assume that we know that:

$$\forall m,n\geq 0:\int_{\mathbb{R}} a_n(x)\,a_m(x)\,d\alpha(x)=\delta_{nm}\hat{a}_n,$$

$$\int_{\mathbb{R}} b_n(x)\,b_m(x)\,d\beta(x)=\delta_{mn}\hat{b}_n,$$

where δ_{mn} denotes as usually Kronecker's delta.

Suppose also that we know connection coefficients between the sets $\{a_n(x)\}_{n\geq 0}$ and $\{b_n(x)\}_{n\geq 0}$ i.e. we know numbers $\gamma_{k,n}$ such that

$$\forall n\geq 1: a_n(x)=\sum_{k=0}^{n} b_k(x)\,\gamma_{k,n}.$$

Further suppose that the measures α and β have densities $A(x)$ and $B(x)$ respectively. Then

$$B(x)=A(x)\sum_{n=0}^{\infty} c_n a_n(x)\,, \tag{2.1}$$

where $c_n=\gamma_{0,n}\hat{b}_0/\hat{a}_n$.

The sense of (2.1) and the type of its convergence depends on the properties of the functions $B(x)$, $A(x)$ and the coefficients $\{c_n\}_{n\geq 1}$. If

$$\int_{\mathbb{R}}\left(B(x)^2/A^2(x)\right)d\alpha(x)<\infty$$

that is if $B(x)/A(x)\in L_2(\mathbb{R},\mathcal{B},\alpha)$, series $\sum_{n=0}^{\infty} c_n a_n(x)$ converges in $L_2(\mathbb{R},\mathcal{B},\alpha)$ and depending on the coefficients $\{c_n\}_{n\geq 0}$ we can even have almost (with respect to α) pointwise convergence (more precisely if $\sum_{i\geq 1}|c_n|^2\log^2 n<\infty$, by the Rademacher–Menshov Thm.).

However in general $B(x)/A(x)$ is only integrable with respect to measure α. Then one has to refer to the distribution theory. $\sum_{n=0}^{\infty} c_n a_n(x)$ is then in general a distribution of order 0.

To see that really

$$c_n=\gamma_{0,n}\hat{b}_0/\hat{a}_n,$$

for $n\geq 0$ let us multiply both sides of (2.1) by $\alpha_n(x)$ and integrate over $\operatorname{supp}\alpha$. On the left hand side we will get $\gamma_{0,n}\hat{b}_0$ since

$$\int_{\mathbb{R}} b_k(x)\,B(x)\,dx=0$$

for $k\geq 1$. On the right hand side we get $c_n\hat{a}_n$.

Remark 1. *Of course to get the expansion (2.1) one needs only to calculate*

$$\int_{\mathbb{R}} \alpha_m(x)\, d\beta(x) = \gamma_{0,m}.$$

On the other hand to get connection coefficients one needs to do some algebra without integration. This sometimes can be simpler.

The idea of relating sets of polynomials given the relationship between measures that make these sets of polynomials orthogonal is not new (see e.g. [12] Thm.2.7.1 (by Christoffel)), assertion iii). Christoffel's relationship between sets of polynomials given the fact that the ratio between orthogonalizing these polynomials measures is a polynomial is accurate given the zeros of this polynomial. If the polynomial is of order more than 2 it is hard to find these zeros as functions of coefficients. This is of course limitation of possible applications of Christoffel's result. The following simple Proposition can be viewed as simplified modification of Christoffel's Theorem. It contains series of simple remarks concerning relationships between discussed sets of polynomials. They do not give precise relationship but in particular situation, confronted together can give such connection. Besides here the only thing one has to know about the ratio of the measures is its expansion with respect to one of these sets of polynomials.

Proposition 1. *Suppose α, β, $A(x)$, $B(x)$, are as described above. Assume also that* $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$. *Suppose further that $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ polynomials are monic*[1]. *Suppose additionally that we know that $B(x)/A(x) = W(x)$,where W can be expanded in the series of polynomials $a_i(x)$:*

$$W(x) = 1 + \sum_{i=1}^{N} w_i a_i(x)/\hat{a}_i$$

where $\hat{a}_i = \int a_i^2(x)\, A(x)\, dx$, converging in $L_2(\mathbb{R},\mathcal{B},\alpha)$. Put $w_0 = 1$. Number N can be finite or infinite. Let us recursively define the sequence of numbers $\{f_n\}_{n\geq 0}$, with $f_0 = 1$ by:

$$n \geq 1 : \sum_{i=0}^{n} f_{n-i} w_i = 0,$$

where we set $w_i = 0$ for $i \geq N+1$ if N is finite.

i) Then monic polynomials defined by:

$$\phi_n(x) = \sum_{i=0}^{n} f_{n-i} a_i(x)$$

satisfy $\int_{\mathbb{R}} \phi_n(x)\, B(x)\, dx = 0$, $n = 1, 2, \ldots$. Besides for $\forall n \geq 1$:

$$a_n(x) = \sum_{i=0}^{n} w_{n-i} \phi_i(x).$$

ii) If N is finite, then $\int a_i(x)\, dB(x) = w_i$, $i = 1, \ldots, N$, and $\int a_i(x)\, dB(x) = 0$, $\forall i \geq N+1$. In particular:

$$a_n(x) = \phi_n(x) + \sum_{i=1}^{N} w_i \phi_{n-i}(x),$$

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[1]Polynomial $p_n(x)$ of order n is called monic if coefficient at x^n is equal to 1.

for $n \geq N+1$.

iii) If N *is finite then there exist* N *sequences* $\left\{\gamma_{n,j}\right\}_{n\geq 1, 1\leq j\leq N}$ *such that* $\forall n\geq 1$

$$a_n\left(x\right)=b_n\left(x\right)+\sum_{j=1}^{N}\gamma_{n,j}b_{n-j}\left(x\right).$$

Proof. Is moved to section 6 □

Remark 2. *The most important assertion of the Proposition above is the assertion iii). It is illustrated by at least two examples presented below: Example 1 where we analyze the ratio of the two densities with respect to which the Chebyshev polynomials of the second and the first kind are orthogonal. This ratio is a polynomial of order* 2 *(* $N=2$*) and thus we have formula (2.3) expressing the Chebyshev polynomials of the first kind as a finite (involving* $3=N+1$ *last only) combination of the Chebyshev polynomials of the second kind. Similar situation is in Example 2, below.*

2.1. **Examples.** Let us denote as usually: $I_A\left(x\right)=\left\{\begin{array}{lll}1 & if & x\in A\\ 0 & if & x\notin A\end{array}\right.$.

(1) Let us take

$$A\left(x\right)=\frac{1}{\pi\sqrt{1-x^2}}I_{\left(-1,1\right)}\left(x\right),\text{and } a_n\left(x\right)=T_n\left(x\right),n\geq -1$$

(Chebyshev Polynomials of the first kind). Further let us take:

$$B\left(x\right)=\frac{2}{\pi}\sqrt{1-x^2}I_{\left(-1,1\right)}\left(x\right),\ b_n\left(x\right)=U_n\left(x\right),n\geq -1$$

(Chebyshev Polynomials of the second kind). It is (see e.g. [1] or [12]) known that

$$\begin{aligned}\int_{-1}^{1}a_n\left(x\right)a_m\left(x\right)A\left(x\right)dx &= \left\{\begin{array}{lll}1 & for & n=m=0\\ \frac{1}{2}\delta_{nm} & for & n\neq 0\text{ or }m\neq 0\end{array}\right.,\\ \int_{-1}^{1}b_n\left(x\right)b_m\left(x\right)B\left(x\right)dx &= \delta_{nm}.\end{aligned}$$

Polynomials $\{T_n\}$ and $\{U_n\}$ satisfy the same three term recurrence however with different initial conditions for $n=1$. Namely $T_{-1}\left(x\right)=U_{-1}\left(x\right)=0$, $T_0\left(x\right)=U_0\left(x\right)=1$, $T_1\left(x\right)=x$, $U_1\left(x\right)=2x$ and

$$2xT_n\left(x\right)=T_{n+1}\left(x\right)+T_{n-1}\left(x\right),\tag{2.2}$$

for $n\geq 0$.

Now notice that

$$\left(U_1\left(x\right)-U_{-1}\left(x\right)\right)/2=x=T_1\left(x\right).$$

Besides we have

$$\begin{aligned}x(U_n\left(x\right)-U_{n-2}\left(x\right))/2&=(U_{n+1}\left(x\right)+U_{n-1}\left(x\right)-U_{n-1}\left(x\right)+U_{n-3}\left(x\right))/2\\&=(U_{n+1}\left(x\right)-U_{n-1}\left(x\right))/2+(U_{n-1}\left(x\right)-U_{n-3}\left(x\right))/2,\end{aligned}$$

which is the three term recurrence (2.2) satisfied by polynomials T_n. Hence:

$$\forall n\geq 1:T_n\left(x\right)=\left(U_n\left(x\right)-U_{n-2}\left(x\right)\right)/2.\tag{2.3}$$

Thus consequently we have $\gamma_{0,0}=1$,

$$\gamma_{k,n}=\left\{\begin{array}{ccc}1/2 & if & k=n\\ -1/2 & if & k=n-2\\ 0 & if & otherwise\end{array}\right. ,$$

for $n\geq 1$. So $\gamma_{0,0}=1$, $\gamma_{0,1}=0,\gamma_{0,2}=-1/2$, $\gamma_{0,n}=0$ for $n\geq 3$. Hence we have elementary relationship

$$B\left(x\right)=A\left(x\right)\left(1-T_{2}\left(x\right)\right)=2A\left(x\right)\left(1-x^{2}\right).$$

Similarly one can deduce that :

$$\forall n\geq 1:U_{n}\left(x\right)=2\sum_{i=0}^{\lfloor n/2\rfloor}T_{n-2i}\left(x\right)-\left(1+\left(-1\right)^{n}\right)/2.$$

Hence $\gamma_{0,2i+1}=0$, $\gamma_{0,2i}=1$, $i=0,1,2,\ldots$. Thus we have

$$A\left(x\right)=B\left(x\right)\sum_{i=0}^{\infty}U_{2i}\left(x\right)$$

and we do not have neither pointwise nor even mod β convergence[2]. One can deduce, following definition of distributions that the right hand side of the above equality is a distribution t_{α} for which $\forall n\geq 1$ $t_{\alpha}\left(T_{n}\right)=0$, by (2.3) and orthogonality of $\left\{U_{i}\right\}_{i\geq 1}$ with respect to $B\left(x\right)$. However we are not going to continue this topic since our main concern are regular, convergent cases. Deeper analysis as well as the generalization of this case can lead to some interesting theoretical problems. In particular what is the meaning of similar expansions in the case when the condition $\operatorname{supp}\beta\subseteq\operatorname{supp}\alpha$ is not satisfied but the connection coefficients are known?

(2) Let

$$A\left(x|y,\rho\right)=\frac{\left(1-\rho^{2}\right)\sqrt{4-x^{2}}}{2\pi((1-\rho^{2})^{2}-\rho xy(1+\rho^{2})+\rho^{2}(x^{2}+y^{2}))}$$

if $x\in(-2,2)$ and 0 otherwise and $|y|\leq 2$, $|\rho|<1$ be a particular case of the Kesten–McKay density considered also in the sequel. It is known (also it follows the fact that it is a particular case of considered in the sequel distribution f_{CN}) that the following polynomials

$$k_{n}\left(x|y,\rho\right)=U_{n}\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)+\rho^{2}U_{n-2}\left(x/2\right)$$

when $n\geq 2$, $k_{1}\left(x|y,\rho\right)=x-\rho y$ and $k_{0}\left(x|y,\rho\right)=1$ are orthogonal with respect to the measure defined by A.

As the measure β let us take same measure as in the previous example but re-scaled by 2. More precisely let β have density

$$B\left(x\right)=\frac{1}{2\pi}\sqrt{4-x^{2}}.$$

Hence re-scaled Chebyshev polynomials $U_{n}\left(x/2\right)$ are orthogonal with respect to β. As far as the expansion of B is concerned we have

$$\gamma_{0,n}=\left\{\begin{array}{ccc}0 & if & n>2\\ \rho^{2} & if & n=2\\ -\rho y & if & n=1\end{array}\right. .$$

[2]'mod β' traditionally in probability means 'in measure β'.

Besides it is known (also from (3.14)) that $\int_{-2}^{2}k_n^2\left(x|y,\rho,0\right)A\left(x|y,\rho\right)=(1-\rho^2)$. Hence we have:

$$\begin{aligned}B\left(x\right) &= A\left(x|y,\rho\right)\left(1-\frac{\rho y}{(1-\rho^2)}k_1\left(x|y,\rho\right)+\frac{\rho^2}{(1-\rho^2)}k_2\left(x|y,\rho\right)\right)\\ &= A\left(x|y,\rho\right)\left((1-\rho^2)^2-\rho(1-q)xy(1+\rho^2)+(1-q)\rho^2(x^2+y^2)\right)/(1-\rho^2).\end{aligned}$$

On the other hand one can easily derive (or it follows from (4.7) in [10] considered for $q=0$ and noting that $h_n\left(x|0\right)=U_n\left(x\right)$) that

$$U_n(x/2)=\sum_{j=0}^{n}\rho^{n-j}U_{n-j}\left(y/2\right)k_j\left(x|y,\rho\right).$$

Thus we have $\gamma_{0,n}=\rho^nU_n\left(y/2\right)$ and consequently:

$$A\left(x|y,\rho\right)=B\left(x\right)\sum_{i=0}^{\infty}\rho^iU_i\left(y/2\right)U_i\left(x/2\right), \tag{2.4}$$

which is a particular case of the Poisson –Mehler kernel to be discussed in the sequel.

(3) Following well known (see e.g. [1] Ex. 5, p. 339) formula concerning Hermite polynomials H_n orthogonal with respect to the measure

$$d\alpha\left(x\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)dx\overset{df}{=}A\left(x\right)dx,$$

$$\forall\rho\in\left(-1,1\right),\forall n\geq 1:H_n\left(\rho x+y\sqrt{1-\rho^2}\right)=\sum_{i=0}^{n}\binom{n}{i}\rho^i\left(\sqrt{1-\rho^2}\right)^{n-i}H_i\left(x\right)H_{n-i}\left(y\right),$$

we can rewrite it in the following form :

$$\forall\rho\in\left(-1,1\right),\forall n\geq 1:H_n\left(x\right)=\sum_{i=0}^{n}\binom{n}{i}\rho^iH_i\left(y\right)\left(\sqrt{1-\rho^2}\right)^{n-i}H_{n-i}\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right),$$

since we have trivially

$$x=\rho y+\sqrt{1-\rho^2}\frac{(x-\rho y)}{\sqrt{1-\rho^2}},$$

and view it as a 'connection coefficient formula' between sets of polynomials $\left\{H_n\left(x\right)\right\}_{n\geq 0}$ that are orthogonal with respect the measure $d\alpha$ and $\left\{\left(\sqrt{1-\rho^2}\right)^nH_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right)\right\}_{n\geq 0}$ that are orthogonal with respect to the measure

$$d\beta\left(x\right)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{\left(x-\rho y\right)^2}{2\left(1-\rho^2\right)}\right)dx\overset{df}{=}B\left(x\right)dx.$$

An easy calculation gives $\gamma_{0,n}=\rho^nH_n\left(y\right)$ and $\hat{a}=n!$ and we end up with famous Mehler Hermite Polynomial Formula

$$\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{\left(x-\rho y\right)^2}{2\left(1-\rho^2\right)}\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\sum_{i=0}^{\infty}\frac{\rho^i}{i!}H_i\left(x\right)H_i\left(y\right), \tag{2.5}$$

which is better known in a form obtained from the above by dividing both sides by $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ and whose proof takes about a page in popular handbooks of special functions like e.g. [1].

3. Densities and families of orthogonal polynomials. Their properties and relationships

3.1. **Notation.** We will use traditional notation of the $q-$series theory i.e.

$$\left[0\right]_q=0;\left[n\right]_q=1+q+\ldots+q^{n-1},\left[n\right]_q!=\prod_{i=1}^{n}\left[i\right]_q,$$

with $\left[0\right]_q!=1$,

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\frac{\left[n\right]_q!}{\left[n-k\right]_q!\left[k\right]_q!} & when \quad n\geq k\geq 0\\ 0 & when \quad otherwise\end{cases}.$$

Sometimes it will be useful to use the so called $q-$Pochhammer symbol :

$$\forall n\geq 1:\left(a;q\right)_n=\prod_{i=0}^{n-1}\left(1-aq^i\right),$$

with $\left(a;q\right)_0=1$,

$$\left(a_1,a_2,\ldots,a_k;q\right)_n=\prod_{i=1}^{k}\left(a_i;q\right).$$

It is easy to notice that $\left(q;q\right)_n=\left(1-q\right)^n\left[n\right]_q!$ and that

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\frac{\left(q;q\right)_n}{\left(q;q\right)_{n-k}\left(q;q\right)_k} & when \quad n\geq k\geq 0\\ 0 & when \quad otherwise\end{cases}.$$

Notice also that $\left(a;0\right)_n=1-a$ for $n\geq 1$ and $\left(a;1\right)_n=\left(1-a\right)^n$.

If it will not cause misunderstanding Pochhammer symbol $(a;q)_n$ or $(a_1,a_2,\ldots,a_k;q)_n$ will often be abbreviated to $\left(a\right)_n$ and $\left(a_1,a_2,\ldots,a_k\right)_n$ if the choice of q is obvious.

Let us also denote :

$$S\left(q\right)=\left[-2/\sqrt{1-q},2/\sqrt{1-q}\right].$$

3.2. **Densities defined by infinite products.** As it follows from the three examples discussed by the end of previous section, the idea of expanding one density with a help of another, can be fruitful and lead to interesting formulae and consequently to deeper understanding of the expanded distribution. Besides some recently used distributions have densities that are defined with a help of infinite products. Infinite products are in many ways difficult to deal with. In particular they are more difficult to calculate many quantities that are interesting for probabilists like moments for example. That is why we will use this technique of expansion to accustom, that is to obtain another, more suitable for further analysis and research, form of the three densities that appeared recently and that are defined by an infinite series.

Two of these three distributions appeared in the context of one dimensional random fields (see details [2] and [4]), $q-$Gaussian processes (for details see [6]) or quadratic harnesses considered by Bryc at al. [18].

All three distributions appeared in the context of special functions in particular in the context of the Rogers polynomials. However only recently their importance

to both commutative and noncommutative probability became apparent. As mentioned before distributions $f_N(x|q)$ and $f_{CN}(x|y,\rho,q)$ that are defined below reappeared in 1997 in the paper [6] of Bożejko and Speicher in a purely noncommutative probability context.

The densities that we are primarily going to analyze are as follows:

$$f_N(x|q) = \frac{\sqrt{1-q}\,(q)_\infty}{2\pi\sqrt{4-(1-q)x^2}}\prod_{k=0}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right) \tag{3.1}$$

defined for $|q|<1$ and $|x|<\frac{2}{\sqrt{1-q}}$ that will be sometimes referred to as $q-$Normal (briefly $q-$N) distribution and

$$f_{CN}(x|y,\rho,q) = \frac{\sqrt{1-q}\left(\rho^2,q\right)_\infty}{2\pi\sqrt{4-(1-q)x^2}}\times \tag{3.2a}$$

$$\prod_{k=0}^{\infty}\frac{\left((1+q^k)^2-(1-q)x^2q^k\right)}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}}, \tag{3.2b}$$

defined for $|q|<1$, $|\rho|<1$, $|x|,|y|<\frac{2}{\sqrt{1-q}}$ that will be referred to as $(y,\rho,q)-$Conditional Normal, (briefly $(y,\rho,q)-CN$) distribution.

The third one it is the so called $q-$utraspherical density (density with respect to which Rogers (called also $q-$utraspherical) polynomials are orthogonal). It is given by

$$f_R(x|\beta,q) = \frac{\sqrt{1-q}\left(\beta^2,q\right)_\infty}{2\pi\sqrt{4-(1-q)x^2}\,(\beta,\beta q)_\infty}\prod_{k=0}^{\infty}\frac{\left((1+q^k)^2-(1-q)x^2q^k\right)}{\left((1+\beta q^k)^2-(1-q)\beta x^2q^k\right)}. \tag{3.3}$$

defined also for $|q|<1$ and $|x|<\frac{2}{\sqrt{1-q}}$ and $|\beta|<1$. This distributions is closely related to distributions $q-$N and $(y,\rho,q)-$CN. Namely we have the following.

Remark 3. *Note that we have*

$$f_R(x|\beta,q) = f_N(x|q)\times\frac{\left(\beta^2\right)_\infty}{(\beta,\beta q)_\infty\prod_{k=0}^{\infty}((1+\beta q^k)^2-(1-q)\beta x^2q^k)}. \tag{3.4}$$

We also have $f_{CN}(x|x,\rho,q) = f_R(x|\rho,q)/(1-\rho)$, *since*

$$(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})x^2+2(1-q)\rho^2x^2q^{2k} = \left(1-\rho q^k\right)^2\left(\left(1+\rho q^k\right)^2-(1-q)\rho x^2q^k\right)$$

and

$$\lim_{\beta\to1^-}f_R(x|\beta,q) = \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}}.$$

3.3. **Polynomials.** Recall that every family of orthogonal polynomials is defined by $3-$ term recursive relationship. The three families of orthogonal polynomials that appear in connection with densities f_N, f_{CN}, f_R are defined by the following recursive relationships:

$$H_{n+1}(x|q) = xH_n(x|q)-[n]_q H_{n-1}(x|q), \tag{3.5}$$

$$\begin{aligned}R_{n+1}(x|\beta,q) &= (1-\beta q^n)\,xR_n(x|\beta,q)\\ &\quad-\left(1-\beta^2q^{n-1}\right)[n]_q R_{n-1}(x|\beta,q),\end{aligned} \tag{3.6}$$

$$\begin{aligned}P_{n+1}(x|y,\rho,q) &= (x-\rho yq^n)P_n(x|y,\rho,q)\\ &\quad-(1-\rho^2q^{n-1})[n]_qP_{n-1}(x|y,\rho,q),\end{aligned} \tag{3.7}$$

with $H_{-1}(x|q) = R_{-1}(x|\beta,q) = P_{-1}(x|y,\rho,q) = 0$, $H_0(x|q) = R_0(x|\beta,q) = P_0(x|y,\rho,q) = 1$.

Here parameters β,ρ,y,q have the following bounds: $|q| \leq 1$, $|\beta|, |\rho| < 1$, $y \in \mathbb{R}$. The family (3.5) will be referred to as the family of $q-$Hermite polynomials, family (3.6) will be referred to as the family of Rogers polynomials. Finally family (3.7) will be referred to as the family of Al-Salam–Chihara polynomials.

In fact in the literature (see e.g. [1]) more popular are these families transformed. Namely as the $q-$Hermite polynomials often function polynomials

$$h_n(x|q) = (1-q)^{n/2} H_n\left(\frac{2x}{\sqrt{1-q}}|q\right), n \geq 1 \tag{3.8}$$

often called also continuous $q-$Hermite polynomials. As Rogers polynomials function polynomials:

$$C_n(x|\beta,q) = (q)_n (1-q)^{n/2} R_n\left(\frac{2x}{\sqrt{1-q}}|\beta,q\right), n \geq 1. \tag{3.9}$$

Finally as Al-Salam–Chihara polynomials function polynomials:

$$p_n(x|a,b,q) = (1-q)^{n/2} P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}}, \sqrt{b}, q\right), \tag{3.10}$$

for $|\beta| < 1, a^2 > b \geq 0$ or even (see e.g. [12]):
(3.11)
$$Q_{n+1}(x|a,b,q) = (2x-(a+b)q^n)Q_n(x|a,b,q) - (1-abq^{n-1})(1-q^n)Q_{n-1}(x|a,b,q),$$

with $Q_{-1}(x|a,b,q) = 0$, $Q_0(x|a,b,q) = 1$ related to polynomials P_n by :

$$P_n(x|y,\rho,q) = Q_n\left(x\sqrt{1-q}/2|\frac{\sqrt{1-q}}{2}\rho(y - i\sqrt{\frac{4}{1-q}-y^2}), \frac{\sqrt{1-q}}{2}\rho(y+i\sqrt{\frac{4}{1-q}-y^2}), q\right)/(1-q)^{n/2}.$$

For our purposes, closely connected with probability, the families defined by (3.5), (3.6) and (3.7) are more suitable.

The families of polynomials $\{H_n\}_{\geq -1}, \{P_n\}_{n\geq -1}$ and $\{R_n\}_{\geq -1}$ have the following basic properties:

Lemma 1. $\forall -1 < q \leq 1, |\rho|, |\beta| < 1, y \in S(q)$ *we have* $\forall n \geq 0$*:*

$$\int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ [n]_q! & when \quad n = m \end{cases}, \tag{3.12}$$

$$\int_{S(q)} H_n(x|q) f_{CN}(x|y,\rho,q)\, dx = \rho^n H_n(y|q), \tag{3.13}$$

(3.14)
$$\int_{S(q)} P_n(x|y,\rho,q) P_m(x|y,\rho,q) f_{CN}(x|y,\rho,q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ (\rho^2)_n [n]_q! & when \quad n = m \end{cases},$$

(3.15)
$$\int_{S(q)} R_n(x|\beta,q) R_n(x|\beta,q) f_R(x|\beta,q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ \frac{(1-\beta)(\beta^2)_n [n]_q!}{(1-\beta q^n)} & when \quad n = m \end{cases},$$

(3.16)
$$\forall \left|\rho_1\right|, \left|\rho_2\right| < 1 : \int_{S(q)} f_{CN}\left(x|y,\rho_1,q\right) f_{CN}\left(y|z,\rho_2,q\right) dy = f_{CN}\left(x|z,\rho_1\rho_2,q\right).$$

$$\max_{x\in S(q)} \left|H_n\left(x|q\right)\right| \leq \frac{W_n\left(q\right)}{(1-q)^{n/2}}, \max_{x\in S(q)} \left|R_n\left(x|\beta,q\right)\right| \leq \frac{V_n\left(q,\beta\right)}{\left(q\right)_n (1-q)^{n/2}}, \tag{3.17}$$

where

$$W_n\left(q\right) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q, V_n\left(q,\beta\right) = \sum_{i=0}^{n} \frac{\left(\beta|q\right)_i \left(\beta|q\right)_{n-i}}{\left(q|q\right)_i \left(q|q\right)_{n-i}}. \tag{3.18}$$

Proof. (3.12), follows from (13.1.11) of [12] after necessary normalization, 3.13 is given in [2], or [4], however can be also deduced from (3.23) below. To prove (3.14) and (3.15) one can use [1] or [12] where these formulae are proved with different normalization (in fact for polynomial Q_n and C_n defined by (3.11 and (3.9 respectively)). Below we show it in an elementary way using standard knowledge on orthogonal polynomials and formulae (3.7) and (3.6). Firstly let us denote

$$\begin{aligned} A_n &= \int_{S(q)} R_n^2(x|\beta,q) f_R\left(x|\beta,q\right) dx, \\ B_n &= \int_{S(q)} P_n^2\left(x|y,\rho,q\right) f_{CN}\left(x|y,\rho,q\right) dx. \end{aligned}$$

Further we multiply both sides (3.7) and (3.6) once respectively by $P_{n-1}\left(x|y,\rho,q\right)$ and $R_{n-1}(x|\beta,q)$ and then by $P_{n+1}\left(x|y,\rho,q\right)$ and $R_{n+1}\left(x|\beta,q\right)$ and integrate respectively with respect. f_{CN} and f_R over $S\left(q\right)$ obtaining respectively

$$\int_{S(q)} xP_n\left(x|y,\rho,q\right) P_{n-1}\left(x|y,\rho,q\right) f_{CN}\left(x|y,\rho,q\right) dx = (1-\rho^2q^{n-1})[n]_q B_{n-1}$$

and

$$(1-\beta q^n) \int_{S(q)} xR_n\left(x|\beta,q\right) R_{n-1}\left(x|\beta,q\right) f_R\left(x|\beta,q\right) dx = \left(1-\beta^2q^{n-1}\right) [n]_q A_{n-1}$$

and then

$$B_{n+1} = \int_{S(q)} xP_{n+1}\left(x|y,\rho,q\right) P_n\left(x|y,\rho,q\right) f_{CN}\left(x|y,\rho,q\right) dx$$

and

$$A_{n+1} = (1-\beta q^n) \int_{S(q)} xR_n\left(x|\beta,q\right) R_{n+1}\left(x|\beta,q\right) f_R\left(x|\beta,q\right) dx.$$

From these equations we deduce that respectively

$$B_n = (1-\rho^2q^{n-1})[n]_q B_{n-1}$$

and

$$A_n = \frac{1-\beta q^{n-1}}{1-\beta q^n} \left(1-\beta^2q^{n-1}\right) [n]_q A_{n-1}$$

from which follows (3.15) and (3.14).

Formula (3.16) is taken from [2] and [3]. It can be also found in [6]. Formula (3.17) follows formulae 13.1.10 and 13.2.16 of [12] and (3.8) and (3.9). □

We will also use the already mentioned Chebyshev polynomials of the first $T_n(x)$ defined by $T_n(\cos\theta) = \cos n\theta$ and second kind $U_n(x)$ defined by $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ and ordinary (probabilistic) Hermite polynomials $H_n(x)$ i.e. polynomials orthogonal with respect to $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$. Recall that Chebyshev polynomials were defined in Subsection 2.1 Example 1 and satisfy $3-$term recurrence (2.2), while polynomials H_n satisfy $3-$term recurrence (3.19) below.

$$xH_n(x) = H_{n+1}(x) + nH_{n-1} \tag{3.19}$$

$H_0(x) = 1$, $H_1(x) = x$. Moreover we will be using re-scaled versions of polynomials T_n and U_n that is

$$\begin{aligned}\hat{T}_n(x|q) &= T_n\left(x\sqrt{1-q}/2\right)/(1-q)^{n/2},\\ \hat{U}_n(x|q) &= U_n\left(x\sqrt{1-q}/2\right)/(1-q)^{n/2}.\end{aligned}$$

These modified polynomials are orthogonal with respect to modified densities that appear in the context of Chebyshev polynomials. That is we have

$$\begin{aligned}\int_{S(q)}\hat{U}_n(x|q)\hat{U}_m(x|q)f_U(x|q)\,dx &= \delta_{mn},\\ \int_{S(q)}\hat{T}_n(x|q)\hat{T}_m(x|q)f_T(x|q)\,dx &= \delta_{mn}/2,\end{aligned}$$

if $n\vee m\geq 1$ and 1 if $n=m=0$, where we denoted

$$f_U(x|q) = I_{S(q)}(x)\sqrt{(1-q)(4-(1-q)x^2)}/2\pi, \tag{3.20}$$

$$f_T(x|q) = I_{S(q)}(x)/(\sqrt{(1-q)/(4-(1-q)x^2)}\pi). \tag{3.21}$$

The density f_U functions sometimes in the literature as the density of Wigner distribution with radius $2/\sqrt{1-q}$ or the density of the semicircle distribution. The density f_T is often called the density of the arcsine distribution.

In the sequel there will also appear distribution $f_{CN}(x|y,\rho,0)$ re-scaled in the following way

$$f_K(x|y,\rho,q) = \frac{\left(1-\rho^2\right)\sqrt{1-q}\sqrt{4-(1-q)x^2}}{2\pi\left(\left(1-\rho^2\right)^2-\rho(1-q)(1+\rho^2)xy+(1-q)\rho^2(x^2+y^2)\right)}I_{S(q)}(x), \tag{3.22}$$

for $-1<q\leq 1$, $|\rho|<1$, $y\in S(q)$, that is a particular case of so called Kesten–McKay distribution and which is nothing else but re-scaled density $A(x)$ considered above in Example 2.

We have Proposition that relates cases defined by special values of parameters to known families of polynomials or distributions:

Proposition 2. 1.

$$f_{CN}(x|y,0,q) = f_R(x|0,q) = f_N(x|q) =$$

$$f_U(x|q)(q)_\infty\times\prod_{k=1}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right),$$

2. $\forall n\geq 0:$

$$R_{n}\left(x|0,q\right)=H_{n}\left(x|q\right),\ H_{n}\left(x|0\right)=U_{n}\left(x/2\right),$$
$$H_{n}\left(x|1\right)=H_{n}\left(x\right),\ \lim_{\beta\longrightarrow1^{-}}\frac{R_{n}\left(x|\beta,q\right)}{\left(\beta\right)_{n}}=2\frac{T_{n}\left(x\sqrt{1-q}/2\right)}{(1-q)^{n/2}},$$

3. $\forall n\geq 0:$

$$\begin{aligned}P_{n}\left(x|x,\rho,q\right) &= R_{n}\left(x|\rho,q\right),\ P_{n}\left(x|y,0,q\right)=H_{n}(x|q),\\ P_{n}(x|y,\rho,1) &= (1-\rho^{2})^{n/2}H_{n}\left(\frac{x-\rho y}{\sqrt{1-\rho^{2}}}\right),\\ P_{n}\left(x|y,\rho,0\right) &= U_{n}\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)+\rho^{2}U_{n-2}\left(x/2\right)\overset{df}{=}k_{n}\left(x|y,\rho\right).\end{aligned}$$

4. *relationship (3.13) reduces for $\rho=0$ to relationship (3.12) with $m=0$,*

5.

$$f_{N}\left(x|0\right)=\frac{1}{2\pi}\sqrt{4-x^{2}}I_{<-2,2>}\left(x\right),\ f_{N}\left(x|1\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^{2}/2\right),\ f_{R}\left(x|1,q\right)=f_{T}\left(x|q\right),$$

6.

$$f_{CN}\left(x|y,\rho,0\right)=f_{K}\left(x|y,\rho\right),\ f_{CN}\left(x|y,\rho,1\right)=\frac{1}{\sqrt{2\pi(1-\rho^{2})}}\exp\left(-\frac{\left(x-\rho y\right)^{2}}{2\left(1-\rho^{2}\right)}\right).$$

Proof. 1. is obvious. 2. follows observation that (3.5) simplifies to (2.2) and (3.19) for $q=0$ and $q=1$ respectively while (3.6) simplifies to (3.5). Value $\lim_{\beta\longrightarrow1^{-}}\frac{R_{n}(x|\beta,q)}{(\beta)_{n}}$ can be found in [12] , formula 13.2.15.
3. First three assertions follow either direct observation in the case of $P_{n}\left(x|y,\rho,0\right)$ or comparison of (3.7) and (3.19) considered for substitution $x\longrightarrow(x-\rho y)/\sqrt{1-\rho^{2}}$ and then multiplication of both sides by $\left(1-\rho^{2}\right)^{(n+1)/2}$ third assertion follows following observations: $P_{-1}\left(x|y,\rho,0\right)=0$, $P_{0}\left(x|y,\rho,0\right)=1$, $P_{1}\left(x|y,\rho,0\right)=x-\rho y$, $P_{2}\left(x|y,\rho,0\right)=x\left(x-\rho y\right)-\left(1-\rho^{2}\right)$, $P_{n+1}\left(x|y,\rho,0\right)=xP_{n}\left(x|y,\rho,0\right)-P_{n-1}\left(x|y,\rho,0\right)$ for $n\geq2$ which is equation (2.2). 5. and 6. Their first assertions are obvious. Secondly we notice that passing to the limit $q\longrightarrow1^{-}$ and applying 2. and 3. we obtain well known relationships defining Hermite polynomials. Hence Hermite polynomials are orthogonal with respect to the measure defined by $f_{N}\left(x|1\right)$. Thus distributions defined by f_{N} and f_{CN} tend to normal $N\left(0,1\right)$ and $N\left(\rho y,\left(1-\rho^{2}\right)\right)$ distributions weakly as $q\longrightarrow1^{-}$. So it is natural to define $f_{N}\left(x|1\right)$ and $f_{CN}\left(x|y,\rho,q\right)$ as they are in 5. and 6. □

As suggested in Proposition 2 we will be using notation $k_{n}\left(x|y,\rho\right)$ instead $P_{n}\left(x|y,\rho,0\right)$ which is simpler. Besides we have $k_{0}\left(x|y,\rho\right)=1$, $k_{1}\left(x|y,\rho\right)=x-\rho y$, $k_{2}\left(x|y,\rho\right)=x(x-\rho y)-(1-\rho^{2})$ and $k_{n+1}\left(x|y,\rho\right)=xk_{n}\left(x|y,\rho\right)-k_{n-1}\left(x|y,\rho\right).$

Remark 4. *Since polynomials $\left\{k_{n}\left(x|y,\rho\right)\right\}_{n\geq0}$ are orthogonal with respect to the measure with density $A\left(x\right)$ of Example 2, or more precisely with density $f_{K}\left(x|y,\rho,0\right)$, we deduce (by simple change of variables in appropriate integral) that polynomials $\left\{k_{n}\left(x\sqrt{1-q}|y\sqrt{1-q},\rho\right)\right\}_{n\geq0}$ are orthogonal with respect to $f_{K}\left(x|y,\rho,q\right)$.*

Hence in particular f_{N} is a generalization of $N\left(0,1\right)$ density, while f_{CN} is a generalization of $N\left(\rho y,1-\rho^{2}\right)$ density. It is also known see e.g. [4] that $f_{CN}\left(x|y,\rho,q\right)/f_{N}\left(x|q\right)$ follows Lancaster type expansion (see e.g. [19]). Namely we have:

$$(3.23)\qquad \prod_{k=0}^{\infty}\frac{(1-\rho^2q^k)}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}} = \sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(x|q)H_n(y|q),$$

converges uniformly and defines the Poisson–Mehler kernel. It is an almost obvious generalization of (2.5) and (2.4). We will prove and generalize it by the expansion idea of this paper in the next section.

4. Auxiliary results

In this section we are going either to recall or to calculate connection coefficients of one family of orthogonal polynomials with respect to the others. First we will recall known results, exposing some of the families of connection coefficients. To do this let us introduce one more family of polynomials $\{B_n\left(x|q\right)\}_{n\geq 0}$ that are orthogonal but with respect to some complex measure. They play an auxiliary role and satisfy the following 3−term recursive equation:

$$(4.1)\qquad B_{n+1}\left(y|q\right) = -q^n yB_n\left(y|q\right) + q^{n-1}\left[n\right]_q B_{n-1}\left(y|q\right); n\geq 0,$$

with $B_{-1}\left(y|q\right)=0$, $B_0\left(y|q\right)=1$. Formula (16) of [4] allows to express them through $q-$Hermite polynomials.

Namely we have: $B_n(x|q) = \begin{cases} i^n q^{n(n-2)/2}H_n(i\sqrt{q}\,x|\frac{1}{q}) & \text{for } q>0 \\ (-1)^{n(n-1)/2}|q|^{n(n-2)/2}H_n(-\sqrt{|q|}\,x|\frac{1}{q}) & \text{for } q<0 \end{cases}$,

where $i=\sqrt{-1}$. Obviously we have $B_n\left(x|0\right)=0$ for $n>2$ and also one can see that $B_n\left(x|1\right)=i^nH_n\left(iy\right),\ n\geq 0$.

The properties of families of polynomials $\{H_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$, including 'connection coefficient formulae' met in the literature, are collected in the following Lemma

Lemma 2. *i)* $\forall n\geq 1: P_n\left(x|y,\rho,q\right)=\sum_{j=0}^{n}\left[\begin{smallmatrix}n\\j\end{smallmatrix}\right]\rho^{n-j}B_{n-j}\left(y|q\right)H_j\left(x|q\right)$,

ii) $\forall n>0: \sum_{j=0}^{n}\left[\begin{smallmatrix}n\\j\end{smallmatrix}\right]B_{n-j}\left(x|q\right)H_j\left(x|q\right)=0$,

iii) $\forall n\geq 0: H_n\left(x|q\right)=\sum_{j=0}^{n}\left[\begin{smallmatrix}n\\j\end{smallmatrix}\right]\rho^{n-j}H_{n-j}\left(y|q\right)P_j\left(x|y,\rho,q\right)$,

iv) $\forall n\geq 0: U_n\left(x\sqrt{1-q}/2\right)=\sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\left[\begin{smallmatrix}n-j\\j\end{smallmatrix}\right]_q H_{n-2j}\left(x|q\right)$

and $H_n\left(y|q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}(1-q)^{-n/2}q^k\left(\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q - q^{n-2k+1}\left[\begin{smallmatrix}n\\k-1\end{smallmatrix}\right]_q\right)U_{n-2k}\left(y\sqrt{1-q}/2\right)$,

v) $\forall n\geq 1, \left|\beta\right|,\left|\gamma\right|<1: R_n\left(x|\gamma,q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}\beta^k\frac{[n]_q!(\gamma/\beta)_k(\gamma)_{n-k}(1-\beta q^{n-2k})}{[k]_q![n-2k]_q!(\beta q)_{n-k}(1-\beta)}R_{n-2k}\left(x|\beta,q\right)$,

in particular: $R_n\left(x|\gamma,q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\gamma^kq^{k(k-1)/2}\frac{[n]_q!(\gamma)_{n-k}}{[k]_q![n-2k]_q!}H_{n-2k}\left(x|q\right)$

and $H_n\left(x|q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}\beta^k\frac{[n]_q!\left(1-\beta q^{n-2k}\right)}{(1-\beta)[k]_q![n-2k]_q!(\beta q)_{n-k}}R_{n-2k}\left(x|\beta,q\right)$.

Proof. Formulae given in assertions i) and ii) are given in Remark 1 following Theorem 1 in [4]. iii) We start with formula (4.7) in [10] that gives connection coefficients of h_n with respect to p_n. Then we pass to polynomials H_n & P_n using formulae $h_n\left(x|q\right)=\left(1-q\right)^{n/2}H_n\left(\frac{2x}{\sqrt{1-q}}|q\right)$, $n\geq 1$ and $p_n(x|a,b,q)=\left(1-q\right)^{n/2}P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}},\sqrt{b},q\right)$. By the way notice that this formula can be

easily derived from assertions i) and ii) by standard change of order of summation. iv) Follows 'change of base' formula in continuous $q-$Hermite polynomials ($i.e.$ polynomials h_n) in e.g. [11], [13] or [14] (formula 7.2) that states that

$$h_n\left(x|p\right)=\sum_{k=0}^{\lfloor n/2\rfloor}c_{n,n-2k}\left(p,q\right)h_{n-2k}\left(x|q\right)$$

where

$$\begin{aligned}c_{n,n-2k}\left(p,q\right) &= \sum_{j=0}^{k}\left(-1\right)^{j}p^{k-j}q^{j(j+1)/2}\begin{bmatrix}n-2k+j\\ j\end{bmatrix}_q\times\\ &\quad(\begin{bmatrix}n\\ k-j\end{bmatrix}_p-p^{n-2k+2j+1}\begin{bmatrix}n\\ k-j-1\end{bmatrix}_p)\end{aligned}$$

again expressed for polynomials h_n, next one observes that $h_n\left(x|0\right)=U_n\left(x\right)$, $\left[\begin{smallmatrix}n\\ k\end{smallmatrix}\right]_0$ $=1$ for $n\geq 0$, $k=0,\ldots,n$ hence we have

$$c_{n,n-2k}\left(0,q\right)=\left(-1\right)^{k}q^{k(k+1)/2}\begin{bmatrix}n-k\\ k\end{bmatrix}_q$$

and consequently

$$U_n\left(x\right)=\sum_{k=0}^{\lfloor n/2\rfloor}\left(-1\right)^{k}q^{k(k+1)/2}\begin{bmatrix}n-k\\ k\end{bmatrix}_q h_{n-2k}\left(x|q\right),$$

similarly we get

$$c_{n,n-2k}\left(q,0\right)=q^{k}(\begin{bmatrix}n\\ k\end{bmatrix}_q-q^{n-2k+1}\begin{bmatrix}n\\ k-1\end{bmatrix}_q)$$

and consequently

$$h_n\left(x|q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}q^{k}(\begin{bmatrix}n\\ k\end{bmatrix}_q-q^{n-2k+1}\begin{bmatrix}n\\ k-1\end{bmatrix}_q)U_{n-2k}\left(x\right).$$

Now it remains to return to polynomials H_n . v) It is in fact the celebrated connection coefficient formula for the Rogers polynomials which was expressed in term of the polynomials C_n (see 13.3.5 of [12]). Other formulae in this assertions are in fact applications of the first formula with $\beta=0$ in the first case and $\gamma=0$ in the second and using the fact that $R_n\left(x|0,q\right)=H_n\left(x|q\right)$. □

We have an important proposition generalizing assertion ii) of the Lemma above. We will use it in the proof of the Lemma 3 below.

Lemma 3. $\forall n\geq 0:$

i)

$$U_n\left(x\sqrt{1-q}/2\right)=\sum_{k=0}^{n}D_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right), \tag{4.2}$$

where

$$\begin{aligned} D_{k,n}\left(y,\rho,q\right) &= \sum_{j=0}^{\lfloor (n-k)/2\rfloor}\left(-1\right)^{j}\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\begin{bmatrix} n-j \\ n-k-j\end{bmatrix}\times \\ &\begin{bmatrix} n-k-j \\ n-k-2j\end{bmatrix}\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right). \end{aligned}$$

ii)

$$k_{n}\left(x\sqrt{1-q}|y\sqrt{1-q},\rho\right)=\sum_{k=0}^{n}C_{k,n}\left(y,\rho,q\right)P_{k}\left(x|y,\rho,q\right), \tag{4.3}$$

where

$$\begin{aligned} C_{k,n}\left(y,\rho,q\right) &= \sum_{j=0}^{\lfloor (n-k)/2\rfloor}\left(-1\right)^{j}\left(1-q\right)^{n/2-j}q^{n-k+j(j-3)/2}\begin{bmatrix} n-1-j \\ n-k-2j\end{bmatrix}_{q}\times \\ &\left(\begin{bmatrix} j+k \\ k\end{bmatrix}_{q}-\rho^{2}q^{k}\begin{bmatrix} j+k-1 \\ k\end{bmatrix}_{q}\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right). \end{aligned}$$

Remark 5. *Notice that*

$$D_{k,n}\left(y,\rho,q\right)\left(\rho^{2}\right)_{k}\left[k\right]_{q}!=\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}}U_{n}\left(x\sqrt{1-q}/2\right)P_{k}\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx$$

and

$$C_{k,n}\left(y,\rho,q\right)\left(\rho^{2}\right)_{k}\left[k\right]_{q}!=\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}}P_{n}\left(x\sqrt{1-q}|y\sqrt{1-q},\rho,0\right)P_{k}\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx.$$

Let us define the following quantity:

$$[2k-1]_{q}!!=\left\{\begin{array}{ccc} 1 & if & k=0 \\ \prod_{i=1}^{k}[2i-1]_{q} & if & k\geq 1\end{array}\right. .$$

We have also some interesting corollaries based on the following easy, elementary observations contained in the Remark below. It is following simple induction applied to formulae (2.2), (3.5), Proposition 2 3. , (4.1), and (3.6).

Remark 6. *i)* $U_{n}\left(0\right)=\left\{\begin{array}{ccc} 0 & if & n=2k-1 \\ (-1)^{k} & if & n=2k\end{array}\right.$, $k=1,2,\ldots$

ii) $U_{n}\left(1\right)=\left(-1\right)^{n}U_{n}\left(-1\right)=\left(n+1\right)$,

iii) $U_{n}\left(\frac{1}{2}\right)=\left(-1\right)^{3\lfloor (n+2)/3\rfloor}\left(\left(n+1-3\left\lfloor (n+2)/3\right\rfloor\right)\right)$,

iv) $H_{n}\left(0|q\right)=\left\{\begin{array}{ccc} 0 & if & n=2k-1 \\ (-1)^{k}\left[2k-1\right]_{q}!! & if & n=2k\end{array}\right.$, $k=1,2\ldots$,

$H_{n}\left(\frac{2}{\sqrt{1-q}}\right)=\frac{W_{n}(q)}{(1-q)^{n/2}}$, *where* $W_{n}\left(q\right)$ *by 3.17 and* $n\geq 1$,

v) $k_{n}\left(0|y,\rho\right)=\left\{\begin{array}{ccc} (-1)^{k}\left(1-\rho^{2}\right) & if & n=2k \\ (-1)^{k-1}\rho y & if & n=2k-1\end{array}\right.$, $k=1,2,\ldots$

$k_{n}\left(1|y,\rho\right)=\left\{\begin{array}{ccc} (-1)^{k}(1-\rho^{2}) & if & n=3k \\ (-1)^{k-1}(-\rho y+\rho^{2}) & if & n=3k-1 \\ (-1)^{k-1}\left(1-\rho y\right) & if & n=3k-2\end{array}\right.$, $k=1,2,\ldots$

vi) $B_n(0|q) = \left\{ \begin{array}{ccc} 0 & if & n = 2k-1 \\ q^{k(k-1)}[2k-1]_q!! & if & n = 2k \end{array} \right., \ k = 1,2,\ldots$

vii) $R_n(0,\beta,q) = \left\{ \begin{array}{ccc} 0 & if & n = 2k-1 \\ (-1)^k \left(\beta^2;q^2\right)_k [2k-1]_q!! & if & n = 2k \end{array} \right..$

Corollary 1. $\forall\rho, q \in (-1,1); n \geq 1$:

i)

$$1 - q^{n(n+1)/2} = \sum_{j=0}^{n-1}(1-q)^{n-j}q^{j(j+1)/2}\begin{bmatrix}2n-j\\ j\end{bmatrix}_q [2n-2j-1]_q!!$$

ii)

$$P_n(0|y,\rho,q) = \sum_{j=0}^{\lfloor n/2 \rfloor}\begin{bmatrix}n\\ 2j\end{bmatrix}_q (-1)^j \rho^{n-2j}B_{n-2j}(y|q)[2j-1]_q!!$$

Proof. i) We put $x = 0$ in Lemma 2 iv), use assertion Remark 6 iv), substitute $n \longrightarrow 2n$, perform necessary simplifications, we get including fact that: $(1-q)^k [2k-1]_q!!$ $= (q|q^2)_{k-1}$ and $H_0(0|q) = 1$ which leads to conclusion that the summand for $j = n$ is equal to $q^{n(n+1)/2}$.

ii) We put $x = 0$ and apply Lemma 2 iii) and then use Remark 6 iv). □

5. Expansions

In this section we are going to apply the general idea of expansion presented in Section 2, use results of Section 4 and obtain expansions of some presented above densities in terms of the others. Since there will be many such expansions to formulate all of them in one theorem would lead to clumsy and unclear statement. Instead we divide this section unto many subsections entitled by the names of the densities that will be discussed in its body.

5.1. f_N **and** f_U**.** Using assertion Lemma 2 iv) we deduce that coefficients $\gamma_{0,n}$ in expanding f_N is given by $\gamma_{0,n} = \left\{ \begin{array}{ccc} 0 & if & n = 2k+1, \\ (-1)^k q^{k(k+1)/2} & if & n = 2k, \end{array} \right. \ k = 0,1,\ldots$ and we end up with an expansion

$$f_N(x|q) = f_U(x|q)\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}U_{2k}\left(x\sqrt{1-q}/2\right), \tag{5.1}$$

which was obtained and discussed in [8] with a help of so called "triple product identity". This formula was recently successfully applied to prove "free infinite divisibility" of the q-Normal (defined above) distribution. For details see [15].

Using another assertion of Lemma 2 iv) we get the reciprocal of the above expansion. Namely we have

$$\gamma_{0,n} = \left\{ \begin{array}{ccc} 0 & if & n = 2k+1, \\ (1-q)^{-k} q^k \left(\begin{bmatrix}2k\\ k\end{bmatrix}_q - q\begin{bmatrix}2k\\ k-1\end{bmatrix}_q\right) & if & n = 2k, \end{array} \right..$$

Notice that $(1-q)^{-k} q^k \left(\begin{bmatrix}2k\\ k\end{bmatrix}_q - q\begin{bmatrix}2k\\ k-1\end{bmatrix}_q\right)/[2k]_q! = \frac{q^k(1-q)^{k+1}}{(q)_k(q)_{k+1}}$. Since we have also (3.12), we get:

$$f_U(x|q) = f_N(x|q)\sum_{k=0}^{\infty}\frac{q^k(1-q)^{k+1}}{(q)_k(q)_{k+1}}H_{2k}(x|q). \tag{5.2}$$

As corollaries we get the following useful formulae that were exposed already in [8] and which are presented here for completeness:

$$(q)_{\infty}\prod_{k=1}^{\infty}\left((1+q^{k})^{2}-(1-q)x^{2}q^{k}\right)=\sum_{k=0}^{\infty}\left(-1\right)^{k}q^{k(k+1)/2}U_{2k}\left(x\sqrt{1-q}/2\right),$$

which reduces (after putting $x=0$) to well known

$$(q)_{\infty}\left(-q\right)_{\infty}^{2}=\left(-q\right)_{\infty}\left(q^{2}|q^{2}\right)_{\infty}=\sum_{k=0}^{\infty}q^{k(k+1)/2}$$

which is a particular case of the 'triple product identity' or (after putting $x^{2}(1-q)$ $=4$) to:

$$(q)_{\infty}^{3}=\sum_{k=0}^{\infty}\left(-1\right)^{k}\left(2k+1\right)q^{k(k+1)/2}.$$

Similarly analyzing (5.2) we get:

$$\prod_{k=1}^{\infty}\left((1+q^{k})^{2}-(1-q)x^{2}q^{k}\right)^{-1}=\sum_{k=0}^{\infty}\frac{q^{k}\left(q^{k+1}\right)_{\infty}\left(1-q\right)^{k}}{\left(q^{2}\right)_{k}}H_{2k}\left(x|q\right),$$

since $\left(q\right)_{\infty}/\left(q\right)_{k}=\left(q^{k+1}\right)_{\infty}$ and $\left(q\right)_{k+1}=\left(1-q\right)\left(q^{2}\right)_{k}$ from which we get for example (by setting $x=0$) identity

$$\frac{1}{\left(q\right)_{\infty}\left(-q\right)_{\infty}^{2}}=1+\sum_{k=1}^{\infty}(-1)^{k}\frac{q^{k}\left(1-q\right)^{k}}{\left(q\right)_{k}\left(q^{2}\right)_{k}}\left[2k-1\right]_{q}!!,$$

or (after inserting $x^{2}\left(1-q\right)=4$ and applying Remark 6 iv)):

$$(q)_{\infty}^{-3}=\sum_{k=0}^{\infty}\frac{q^{k}W_{2k}\left(q\right)}{\left(q\right)_{k}\left(q^{2}\right)_{k}}.$$

5.2. f_{N} **and** f_{CN}**.** We use Lemma 2 i) we deduce that coefficients $\gamma_{0,n}$ in expanding f_{CN} are given by $\gamma_{0,n}=\rho^{n}B_{n}\left(y|q\right).$ Keeping in mind (3.14) we get

$$f_{N}\left(x|q\right)=f_{CN}\left(x|y,\rho,q\right)\sum_{n=0}^{\infty}\frac{\rho^{n}}{\left(\rho^{2}\right)_{n}\left[n\right]_{q}!}B_{n}\left(y|q\right)P_{n}\left(x|y,\rho,q\right). \tag{5.3}$$

We use Lemma 2 iii) we deduce that coefficients $\gamma_{0,n}$ in expanding f_{CN} is given by $\gamma_{0,n}=\rho^{n}H_{n}\left(y|q\right).$ Keeping in mind (3.12) we get:

$$f_{CN}\left(x|y,\rho,q\right)=f_{N}\left(x|q\right)\sum_{n=0}^{\infty}\frac{\rho^{n}}{\left[n\right]_{q}!}H_{n}\left(y|q\right)H_{n}\left(x|q\right). \tag{5.4}$$

Notice that (5.4) it is in fact the famous Poisson–Mehler kernel of the $q-$ Hermite polynomials, while (5.3) is its reciprocal. Compare [5] for another proof of (5.4). Notice that for every fixed m, $\sum_{n=0}^{m}\frac{\rho^{n}}{\left(\rho^{2}\right)_{n}\left[n\right]_{q}!}B_{n}\left(y|q\right)P_{n}\left(x|y,\rho,q\right)$ is not a symmetric function of x and y, while when $m=\infty$ it is!

As a corollary (after putting $y=x$ and then using Remark 3) we get the following interesting expansion

$$\frac{\left(\rho^{2}\right)_{\infty}}{\left(\rho\right)_{\infty}^{2}\prod_{k=0}^{\infty}\left(\left(1+\rho q^{k}\right)^{2}-\left(1-q\right)\rho x^{2}q^{k}\right)}=\sum_{n=0}^{\infty}\frac{\rho^{n}}{\left[n\right]_{q}!}H_{n}^{2}\left(x|q\right), \tag{5.5}$$

which reduces to the well known formula (see [12], Exercise 12.3(b))

$$\frac{\left(\rho^2\right)_\infty}{(\rho)_\infty^4}=\sum_{n=0}^{\infty}\frac{\rho^n}{(q)_n}W_n^2(q),$$

after inserting $x=2/\sqrt{1-q}$ and applying 3.17 with W_n defined by (3.18). Expansion (5.5) after inserting $x=0$, can be reduced to:

$$\prod_{k=0}^{\infty}\frac{\left(1-\rho^2q^{2k+1}\right)}{\left(1-\rho^2q^{2k}\right)}=1+\sum_{k=1}^{\infty}\rho^{2k}\prod_{j=1}^{k}\frac{\left(1-q^{2j-1}\right)}{\left(1-q^{2j}\right)},$$

since as it can be easily noticed $\frac{\left([2k-1]_q!!\right)^2}{[2k]_q!}=\prod_{j=1}^{k}\frac{\left(1-q^{2j-1}\right)}{\left(1-q^{2j}\right)}$ and $\frac{\left(\rho^2\right)_\infty}{(\rho)_\infty^2(-\rho^2)_\infty^2}=\prod_{k=0}^{\infty}\frac{\left(1-\rho^2q^{2k+1}\right)}{\left(1-\rho^2q^{2k}\right)}$.

As far as convergence of series (5.3) and (5.4) is concerned then we see that for $|\rho|,|q|<1$ and $x,y\in S_q$ function $g(x|y,\rho,q)=f_{CN}(x|y,\rho,q)/f_N(x|q)=(\rho_2)_\infty\prod_{k=0}^{\infty}\frac{1}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}}$ both bounded and 'cut away from zero' hence its square as well as reciprocal of this square are integrable on compact interval S_q. For exact bounds see [9] Proposition 1 vii).

Remark 7. *Dividing both sides of (5.3) and (5.4) by $f_N(x|q)$, letting $q\longrightarrow 1^-$ and keeping in mind that $B_n(x|1)=i^nH_n(ix)$ and that $P_n(x|y,\rho,1)=\left(\sqrt{1-\rho^2}\right)^nH_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right)$ we get:*

$$1/\sum_{n=0}^{\infty}\frac{\rho^n}{n!}H_n(x)H_n(y)=\sum_{n=0}^{\infty}\frac{\rho^ni^n}{n!\left(1-\rho^2\right)^{n/2}}H_n(ix)H_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right) \tag{5.6}$$

Here however situation is different. The series $\sum_{n=0}^{\infty}\frac{\rho^n}{n!}H_n(x)H_n(y)$, as it is known, is convergent for all $x,y\in\mathbb{R}$ and $|\rho|<1$, while the series (5.6) only for $x,y\in\mathbb{R}$ and $\rho^2<1/2$ since only then the function $f_N^2(x|q)/f_{CN}(x|y,\rho,q)=\exp(-\frac{(x-\rho y)^2}{2(1-\rho^2)}+x^2)$ is integrable with respect to x over whole $\mathbb{R}$.

5.3. f_N **and** f_R**.** We use the last two statements of Lemma 2 v). We deduce that coefficients $\gamma_{0,n}$ in expanding f_R are given by

$$\gamma_{0,n}=\begin{cases}0 & if\quad n=2k+1,\\ \frac{[2k]_q!\beta^k}{[k]_q!(\beta q)_k} & if\quad n=2k,\end{cases},$$

$k=0,1,\ldots$. Keeping in mind (3.12) we get:

$$f_R(x|\beta,q)=f_N(x|q)\sum_{k=0}^{\infty}\frac{\beta^k}{[k]_q!\,(\beta q)_k}H_{2k}(x|q). \tag{5.7}$$

As a corollary let us take $\beta=\rho$ and use (3.4) and compare it with (5.5). We will get then for $|q|,|\rho|<1,x^2(1-q)\leq 2$:

$$(1-\rho)\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n^2(x)=\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!\,(\rho q)_n}H_{2n}(x|q).$$

Next we use second assertion of v) of Lemma 2 and deduce that coefficient $\gamma_{0,n}$ in expanding f_N is by

$$\gamma_{0,n}=\left\{\begin{array}{ccc} 0 & if & n=2k+1, \\ \left(-\gamma\right)^k q^{k(k-1)/2}\frac{[2k]_q!(\gamma)_k}{[k]_q!} & if & n=2k, \end{array}\right. ,$$

$k=0,1,\ldots$. We use also (3.15) and get

$$f_N\left(x|q\right)=f_R\left(x|\gamma,q\right)\sum_{k=0}^{\infty}\left(-\gamma\right)^k q^{k(k-1)/2}\frac{(\gamma)_k\left(1-\gamma q^{2k}\right)}{(1-\gamma)\left[k\right]_q!\left(\gamma^2\right)_{2k}}R_{2k}\left(x|\gamma,q\right). \tag{5.8}$$

Again we can deduce that one of the series (5.8) and (5.7) is the reciprocal of the other.

5.4. f_K **and** f_{CN}**.** Recall that the densities f_K and f_{CN} are given by (3.22) and (3.2) respectively. We will be using Lemma 3 ii) , Remark 4, and the fact that for $n\geq 1:$

$$\begin{aligned} &\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}}f_K\left(\xi|y,\rho,q\right)k_n^2\left(\xi\sqrt{1-q}|y\sqrt{1-q},\rho\right)d\xi \\ =\ &\frac{1}{\sqrt{1-q}}\int_{-2}^{2}f_K\left(x/\sqrt{1-q}|y/\sqrt{1-q},\rho,0\right)k_n^2\left(x|y,\rho\right)dx=\frac{\left(1-\rho^2\right)}{\sqrt{1-q}}. \end{aligned}$$

Beside notice that $C_{0,1}\left(y,\rho,q\right)=1$. Hence $\beta_1\left(y,\rho,q\right)=0$. Consequently we get $\forall x\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;y\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;0<|\rho|<1;q\in(-1,1)$

$$f_{CN}\left(x|y,\rho,q\right)=f_K\left(x|\rho,q\right)(1+\sum_{n=2}^{\infty}\beta_n\left(y,\rho,q\right)k_n(x\sqrt{1-q}|y\sqrt{1-q},\rho))),$$

where $\beta_k\left(y,\rho,q\right)=\sum_{j=1}^{\lfloor k/2\rfloor}(-1)^j\left(1-q\right)^{k/2-j}q^{k+j(j-3)/2}\left[\begin{smallmatrix}k-1-j\\k-2j\end{smallmatrix}\right]\rho^{k-2j}H_{k-2j}\left(y|q\right).$

5.5. f_U **and** f_{CN}**.** Using Lemma 3 i) and calculating in the similar way we get: $\forall x\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;y\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;0<|\rho|<1;q\in(-1,1),$

$$f_{CN}\left(x|y,\rho,q\right)=f_U\left(x|q\right)(1+\sum_{k=1}^{\infty}\gamma_k\left(y,\rho,q\right)U_k\left(x\sqrt{1-q}/2\right)), \tag{5.9}$$

with $\gamma_k\left(y,\rho,q\right)=\sum_{j=0}^{\lfloor k/2\rfloor}\left(-1\right)^j\left(1-q\right)^{k/2-j}\times q^{j(j+1)/2}\left[\begin{smallmatrix}k-j\\k-2j\end{smallmatrix}\right]_q\rho^{k-2j}H_{k-2j}\left(y|q\right).$

Corollary 2.

$$\begin{aligned} &\left(q^3;q^3\right)_\infty\sum_{k=0}^{\infty}\frac{\left(1-q\right)^{k/2}\rho^k}{\left(q\right)_k}H_k\left(y|q\right)\eta_k\left(q\right) \\ =\ &\frac{\left(\rho^2\right)_\infty\left(q^3;q^3\right)_\infty}{\prod_{k=0}^{\infty}\left(1+\rho^2q^{2k}+\rho^4q^{4k}-\sqrt{1-q}\rho yq^k(1+\rho^2q^{2k})+(1-q)\rho^2y^2q^{2k}\right)} \\ =\ &1+\sum_{k=1}^{\infty}\left(-1\right)^{3k}\left(\gamma_{3k}\left(y,\rho,q\right)+\gamma_{3k+1}\left(y,\rho,q\right)\right), \end{aligned}$$

where $\left\{\eta_k\left(q\right)\right\}_{k\geq-1}$ *are given recursively* $\eta_{-1}\left(q\right)=0,\eta_0\left(q\right)=1,\ \eta_{k+1}\left(q\right)=\eta_k\left(q\right)-\left(1-q^k\right)\eta_{k-1}\left(q\right),\ k\geq 0.$

Proof. We insert $x = 1/\sqrt{1-q}$ in (5.9) and use Remark 6 iii) which simplifies to simple rule $U_{3m+2}(1/2) = 0, U_{3m}(1/2) = U_{3m+1}(1/2)(-1)^{3m}$. Then we insert $x = 1/\sqrt{1-q}$ in (3.2) and (3.20) and use the fact that $(1-\rho^2 q^{2k})^2 + \rho^2 q^{2k} = 1 + \rho^2 q^{2k} + \rho^4 q^{4k}$. On the way we also use (3.23), identity $(q)_\infty \prod_{k=1}^{\infty}\left(1+q^k+q^{2k}\right)$ $= \prod_{k=1}^{\infty}\left(1-q^{3k}\right) = \left(q^3;q^3\right)_\infty$, the fact $(1-q)^{k/2} H_k\left(\frac{1}{\sqrt{1-q}}\right) = h_k(1/2)$ and the fact the continuous $q-$Hermite polynomials $h_n(x|q)$ satisfy relationship: $h_{n+1}(x|q)$ $= 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q)$. □

6. Proofs

Let us start this section with very brief recollection of basic facts concerning orthogonal polynomials.

(1) If $\{D_n(x)\}_{n\geq 0}$ is a sequence of polynomials with respect to certain signed measure, then $\{\eta_n D_n(x)\}_{n\geq 0}$, for any nonzero sequence of reals $\{\eta_n\}$ has the same property. Thus we can consider only monic sequences of orthogonal polynomials.
(2) Every monic sequence of orthogonal polynomials say $\{D_n(x)\}_{n\geq 0}$ satisfies the so called three term recurrence (3TR) that is there exist two sequences of reals $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ such that for every $n\geq 0$ we have
$$xD_n(x) = D_{n+1}(x) + \alpha_n D_n(x) + \beta_n D_{n-1}(x),$$
with $D_{-1}(x) = 0, D_0(x) = 1$.
(3) More over when we have a sequence of monic polynomials that satisfies some 3TR with given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ then there exists at least one signed measure such that these polynomials are orthogonal with respect this measure. This statement functions in the literature as "Favard's Theorem".
(4) If $n\geq 0$ we have $\beta_n > 0$ then this signed measure is a positive measure.
(5) There exists more subtle conditions imposed on sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that guarantee that the orthogonalizing measure is unique or that it has the density.

For details see [17], [12] or [16].

Proof of the Proposition 1. i) Notice that ϕ_n is a monic polynomial of degree n, for $n\geq 1$. Now let us calculate $\int_{\mathbb{R}} \phi_n(x) B(x)\,dx$. We have :
$$\begin{aligned}\int_{\mathbb{R}} \phi_n(x) B(x)\,dx &= \sum_{i=0}^{n}\sum_{j=0}^{N} f_{n-i}\frac{w_j}{\hat{a}_j}\int_{\mathbb{R}} a_i(x)\,a_j(x)\,A(x)\,dx \\ &= \sum_{i=0}^{n} f_{n-i}w_i = 0\end{aligned}$$
for $n\geq 1$. Conversely, let us consider polynomial defined by $p_n(x) = \sum_{i=0}^{n} w_{n-i}\phi_i(x)$. We have $p_n(x) = \sum_{i=0}^{n} w_{n-i}\sum_{j=0}^{i} f_{i-j}a_j(x) = \sum_{j=0}^{n} a_j(x)\sum_{i=j}^{n} w_{n-i}f_{i-j} =$ $\sum_{j=0}^{n} a_j(x)\sum_{k=0}^{n-j} w_{n-j-k}f_k = \sum_{j=0}^{n} a_j(x)\sum_{s=0}^{n-j} w_s f_{n-j-s} = a_n(x)$.

ii) Let $i\leq N$. Keeping in mind representation of $W(x)$ and orthogonality of polynomials $a_i(x)$ with respect to the measure α we get.
$$\int_{\mathbb{R}} a_i(x)\,B(x)\,dx = \int_{\mathbb{R}} a_i(x)\,W(x)\,A(x)\,dx = w_i.$$

Similarly if $i>N$ we get zero by the orthogonality of $\left\{a_i\right\}_{i\geq 0}$with respect to $A\left(x\right)$.

iii) Let us define coefficients $c_{n,i}$ by the following expansion:

$$a_n\left(x\right)=\sum_{i=0}^{n}c_{n,i}b_i\left(x\right),$$

The fact that $\left\{a_n\right\}$ and $\left\{b_n\right\}$ are monic implies that $\forall n\geq 0:c_{n,n}=1$. ii) implies that $c_{i,0}=w_i,\ i\leq n;c_{n,0}=0$ for $n\geq N+1$. Besides we have the following relationships between coefficients $c_{n,i}$ that is implied by $3-$terms recurrences satisfied by families $\left\{a_i\right\}$ and $\left\{b_i\right\}$. On one hand we have $xa_n\left(x\right)=a_{n+1}\left(x\right)+\alpha_na_n\left(x\right)+\hat{\alpha}_na_{n-1}\left(x\right)=b_{n+1}\left(x\right)+(\alpha_n+c_{n+1,n})b_n\left(x\right)+\sum_{i=0}^{n-1}\left(c_{n+1,i}+\alpha_nc_{n,i}+\hat{\alpha}_nc_{n-1,i}\right)b_i\left(x\right)$ on the other $xa_n\left(x\right)=\sum_{i=0}^{n}c_{n,i}(b_{i+1}\left(x\right)+\beta_ib_i\left(x\right)+\hat{\beta}_ib_{i-1}\left(x\right))=b_{n+1}\left(x\right)+(c_{n,n-1}+\beta_n)b_n\left(x\right)+\sum_{i=1}^{n-1}(c_{n,i-1}+\beta_ic_{n,i}+\hat{\beta}_ic_{n,i+1})b_i\left(x\right)+\beta_0c_{n,0}+\hat{\beta}_1c_{n,1}$. Equating these two sides we get:

$$\alpha_n+c_{n+1,n}=c_{n,n-1}+\beta_n,$$
$$\forall 1\leq i\leq n-1:c_{n+1,i}+\alpha_nc_{n,i}+\hat{\alpha}_nc_{n-1,i}=c_{n,i-1}+\beta_ic_{n,i}+\hat{\beta}_ic_{n,i+1},$$
$$c_{n+1,0}+\alpha_nc_{n,0}+\hat{\alpha}_nc_{n-1,0}=\beta_0c_{n,0}+\hat{\beta}_1c_{n,1}.$$

From the last of these equations we deduce that $c_{n,1}=0$ for $n\geq N+2$. Similarly by considering equation

$$c_{n+1,1}+\alpha_nc_{n,1}+\hat{\alpha}_nc_{n-1,1}=c_{n,0}+\beta_ic_{n,1}+\hat{\beta}_ic_{n,2}$$

we deduce that $c_{n,2}=0$ for $n\geq N+3$ and so on. We see that then $c_{n,i}=0$ for $n\geq N+i+1$. In particular it means that $c_{n,n-j}=0$ for $j\geq N+1$. □

Proof. of Lemma 3. i) We will argue straightforwardly using Lemma 2 *i*) and *ii*) and then comparing it with assertion iv) of the same Lemma.

We have
$\sum_{k=0}^{n}D_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right)=\sum_{k=0}^{n}D_{k,n}\left(y,\rho,q\right)\sum_{i=0}^{k}\left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right]_q\rho^{k-i}B_{k-i}\left(y|q\right)H_i\left(x|q\right)$
$=$
$\sum_{i=0}^{n}H_i\left(x|q\right)\sum_{k=i}^{n}\left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right]_qD_{k,n}\left(y,\rho,q\right)B_{k-i}\left(y|q\right)$. Let us denote
$G_{i,n}\left(y,\rho,q\right)=\sum_{k=i}^{n}\left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right]_qD_{k,n}\left(y,\rho,q\right)B_{k-i}\left(y|q\right)$. We have using formula for $D_{k,n}\left(y,\rho,q\right)$.

$$\begin{aligned}G_{i,n}(y,\rho,q)&=\sum_{k=i}^{n}\begin{bmatrix}k\\ i\end{bmatrix}_q\rho^{k-i}B_{k-i}\left(y|q\right)\times D_{k,n}\left(y,\rho,q\right)\\&=\sum_{k=i}^{n}\begin{bmatrix}k\\ i\end{bmatrix}_q\rho^{k-i}B_{k-i}\left(y|q\right)\sum_{j=0}^{\lfloor(n-k)/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\begin{bmatrix}n-j\\ n-k-j\end{bmatrix}\\&\quad\times\begin{bmatrix}n-k-j\\ n-k-2j\end{bmatrix}\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)\\&=\sum_{j=0}^{\lfloor(n-i)/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\rho^{n-i-2j}\begin{bmatrix}n-j\\ j\end{bmatrix}_q\begin{bmatrix}n-2j\\ i\end{bmatrix}_q\times\\&\quad\sum_{k=i}^{n-2j}\begin{bmatrix}n-i-2j\\ k-i\end{bmatrix}_qB_{k-i}\left(y|q\right)H_{n-k-2j}\left(y|q\right).\end{aligned}$$

Now $\sum_{k=i}^{n-2j}\left[\begin{smallmatrix}n-i-2j\\k-i\end{smallmatrix}\right]_qB_{k-i}\left(y|q\right)H_{n-k-2j}\left(y|q\right)=$ $\sum_{s=0}^{n-i-2j}\left[\begin{smallmatrix}n-i-2j\\s\end{smallmatrix}\right]_qB_s\left(y|q\right)H_{n-i-2j-s}\left(y|q\right)=\left\{\begin{array}{ccc}1 & if & n-i=2j\\0 & if & n-i>2j\end{array}\right.$ by Lemma 2 ii).

Hence $G_{i,n}\left(y,\rho,q\right)=\left\{\begin{array}{ccc}0 & if & n-i \text{ is odd}\\(-1)^m(1-q)^{n/2-m}q^{m(m+1)/2}\left[\begin{smallmatrix}n-m\\m\end{smallmatrix}\right]_q & if & n-i=2m\end{array}\right.$.

So $\sum_{k=0}^{n}D_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right)=\sum_{i=0}^{n}H_i\left(x|q\right)G_{i,n}\left(y,\rho,q\right)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m(1-q)^{n/2-m}q^{m(m+1)/2}\left[\begin{smallmatrix}n-m\\m\end{smallmatrix}\right]_qH_{n-2m}\left(x|q\right)=U\left(x\sqrt{1-q}/2\right)$ Lemma 2 iv).

ii) Notice that $C_{0,0}\left(y,\rho,q\right)=1$, $C_{n,n}\left(y,\rho,q\right)=(1-q)^{n/2}$, $C_{0,n}\left(y,\rho,q\right)=(1-\rho^2)\times\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(1-q)^{n/2-j}q^{n+j(j-3)/2}\times\left[\begin{smallmatrix}n-1-j\\j\end{smallmatrix}\right]_q\rho^{n-2j}H_{n-2j}\left(y|q\right)$, $C_{n-1,n}\left(y,\rho,q\right)=(1-q)^{n/2}q\rho y\left[n-1\right]_q$. Hence $C_{0,1}(y,\rho,q)=0$ and $C_{1,1}(y,\rho,q)=(1-q)^{1/2}$, $C_{0,2}\left(y,\rho,q\right)=-(1-\rho^2)$, $C_{1,2}\left(y,\rho,q\right)=(1-q)q\rho y$. Thus equation (4.3) is satisfied for $n=0,1,2$. For larger n formula will be proved straightforwardly. Let us consider an expression $W_n\left(x|y,\rho,q\right)=\sum_{k=0}^{n}C_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right)$. We have

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= \sum_{k=0}^{n}P_k\left(x|y,\rho,q\right)\sum_{j=0}^{\lfloor(n-k)/2\rfloor}(-1)^j\left(1-q\right)^{n/2-j}q^{n-k+j(j-3)/2}\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\\&\times\left(\begin{bmatrix}j+k\\k\end{bmatrix}_q-\rho^2q^k\begin{bmatrix}j+k-1\\k\end{bmatrix}_q\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)\\&= \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\sum_{k=0}^{n-2j}\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\\&\times\left(\begin{bmatrix}j+k\\k\end{bmatrix}_q-\rho^2q^k\begin{bmatrix}j+k-1\\k\end{bmatrix}_q\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\end{aligned}$$

Now $n-k+j\left(j-3\right)/2=j\left(j+1\right)/2+n-k-2j$, $\left[\begin{smallmatrix}n-1-j\\n-k-2j\end{smallmatrix}\right]_q\left[\begin{smallmatrix}j+k\\k\end{smallmatrix}\right]_q=\frac{[n-1-j]_q![j+k]_q}{[n-k-2j]_q![k]_q![j]_q!}=\frac{[j+k]_q}{[n-j]_q}\left[\begin{smallmatrix}n-j\\j\end{smallmatrix}\right]_q\left[\begin{smallmatrix}n-2j\\k\end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix}n-1-j\\n-k-2j\end{smallmatrix}\right]_q\left[\begin{smallmatrix}j+k-1\\k\end{smallmatrix}\right]_q=\frac{[n-1-j]_q!}{[n-k-2j]_q![k]_q![j-1]_q!}=\left[\begin{smallmatrix}n-1-j\\j-1\end{smallmatrix}\right]_q\left[\begin{smallmatrix}n-2j\\k\end{smallmatrix}\right]_q$, hence

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\frac{1}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q\\&\times\sum_{k=0}^{n-2j}\begin{bmatrix}n-2j\\k\end{bmatrix}_qq^{n-k-2j}\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\\&-\rho^2\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}q^{n-2j}\begin{bmatrix}n-1-j\\j-1\end{bmatrix}_q\\&\times\sum_{k=0}^{n-2j}\begin{bmatrix}n-2j\\k\end{bmatrix}_q\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right).\end{aligned}$$

Now we apply Lemma 2 iii) and also the simple fact that $q^{n-k-2j}[k+j]_q=[n-j]_q$ $-[n-k-2j]_q$. We get after applying Lemma 2 iv)

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= U_n\left(x\sqrt{1-q}/2\right)-\sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q\\&\times\sum_{k=0}^{n-2j-1}\begin{bmatrix}n-2j-1\\k\end{bmatrix}_q\rho^{n-2j-k}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\\&-\rho^2\sum_{j=1}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}q^{n-2j}\begin{bmatrix}n-1-j\\j-1\end{bmatrix}_q H_{n-2j}\left(x|q\right).\end{aligned}$$

Now we apply formula $H_{n-k-2j}\left(y|q\right)=yH_{n-1-k-2j}(y|q)-[n-1-2j-k]_qH_{n-2-2j-k}(y|q)$ and split the first sum into two. Since $\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q=\begin{bmatrix}n-1-j\\j\end{bmatrix}_q$ we see that the first of these two sums is equal to $\rho\sqrt{1-q}yU_{n-1}\left(x\sqrt{1-q}/2\right)$. Hence

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= U_n\left(x\sqrt{1-q}/2\right)-\rho\sqrt{1-q}yU_{n-1}\left(x\sqrt{1-q}/2\right)\\&+\sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q\\&\times\sum_{k=0}^{n-2j-1}\begin{bmatrix}n-2j-1\\k\end{bmatrix}_q[n-1-k-2j]_q\rho^{n-2j-k}H_{n-2-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\\&+\rho^2\sum_{j=0}^{\lfloor n/2-1\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-1-j}q^{j(j+1)/2}q^{n-j-1}\begin{bmatrix}n-2-j\\j\end{bmatrix}_q H_{n-2-2j}\left(x|q\right).\end{aligned}$$

Notice that

$$\sum_{k=0}^{n-2j-1}\begin{bmatrix}n-2j-1\\k\end{bmatrix}_q[n-1-k-2j]_q\rho^{n-2j-k}H_{n-2-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)=[n-1-2j]_q\rho^2H_{n-2-2j}\left(x|q\right)$$

by Lemma 2 iii). Besides $\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q=\begin{bmatrix}n-1-j\\j\end{bmatrix}_q$. Thus the sum of the last two summands is equal to

$$\rho^2(1-q)\sum_{j=0}^{\lfloor n/2\rfloor-1}\left(-1\right)^j\left(1-q\right)^{n/2-1-j}q^{j(j+1)/2}\begin{bmatrix}n-1-j\\j\end{bmatrix}_q\left[n-1-2j\right]_q H_{n-2-2j}\left(x|q\right)+$$
$$\rho^2\sum_{j=0}^{\lfloor n/2-1\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-1-j}q^{j(j+1)/2}q^{n-j-1}\begin{bmatrix}n-2-j\\j\end{bmatrix}_q H_{n-2-2j}\left(x|q\right).$$

Now

$$\begin{bmatrix}n-1-j\\j\end{bmatrix}_q\left[n-1-2j\right]_q=\left[n-1-j\right]_q\begin{bmatrix}n-2-2j\\j\end{bmatrix}_q$$

and $\left(1-q\right)\left[n-1-j\right]=1-q^{n-1-j}$, hence the sum of last two summands is equal to

$$\rho^2\sum_{j=0}^{\lfloor n/2-1\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-1-j}q^{j(j+1)/2}\begin{bmatrix}n-2-j\\j\end{bmatrix}_q H_{n-2-2j}\left(x|q\right)=\rho^2U_{n-2}\left(x\sqrt{1-q}/2\right)$$

by Lemma 2 iv) □

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